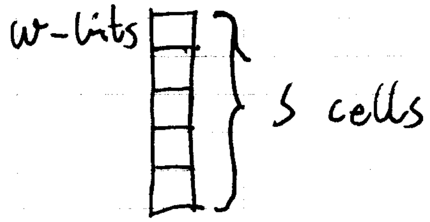
Static Datastructures

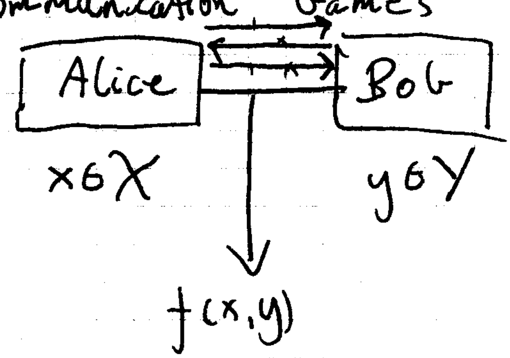
preproces → space S



query$(x) \to \{0,1\}$

↳ 1 cell

Communication Games



(Measuring bits from Alice to Bob and vice versa)

Alice sends a bits

Bob —"— b —"—

$$\text{Alice (query } x) \underset{w}{\overset{\lg S}{\rightleftarrows}} \text{Bob (database)}$$

cell probe complexity: t

$a = t \cdot \lg S$

$b = t \cdot w$

Example:

$\{0,1\}^d \leftarrow n$ points in

query $x \in \{0,1\}^d$

$\lambda \in [0,1]$

Theorem:

Either $a = \Omega(d)$

or $b = \Omega(n^{1-\varepsilon}) \quad \forall \varepsilon > 0$

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$$t \lg S = \Omega(d) \Rightarrow t = \Omega\left(\frac{d}{\lg S}\right) \quad \left(t = \Omega\left(\frac{d}{\lg n}\right) \quad S = n^{O(1)}\right)$$

$$t w = \Omega(n^{1-\varepsilon}) \Rightarrow t = \Omega(n^{1-\varepsilon})$$

$$w = \lg^{O(1)} n$$

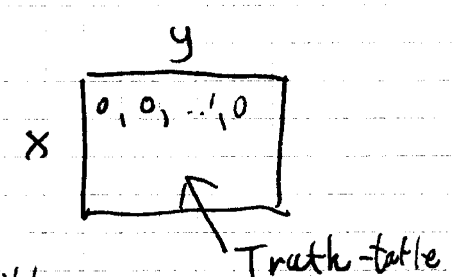
Richness

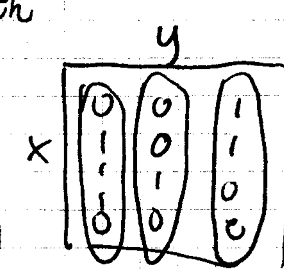
Def:

f is $[u,v]$-rich if

truth-table has v columns with

u ones





f balanced. Density $1s = \Omega(1)$

$\Downarrow$

$[\Omega(|X|), \Omega(\ )]$ - rich

Lemma (Richness)

$\exists$ comm. protocol w a, b

f is $[u,v]$-rich

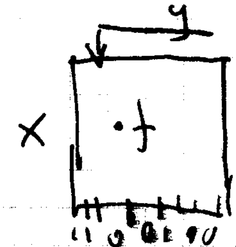
$\Downarrow$

$\exists$ rectangle of f of size $\frac{u}{2^{O(a)}} \cdot \frac{v}{2^{(b)}}$

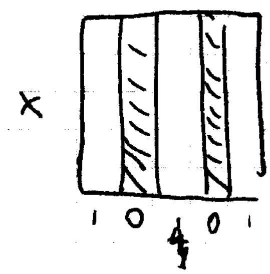
(Rectangle: $X \subset \mathcal{X}$, $Y \subset \mathcal{Y}$, $X \times Y$ is rectangle) proof

Induction on protocol

- Bob sends first bit

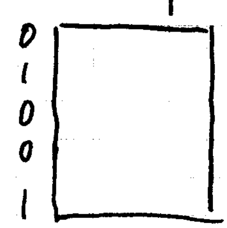


E.g. $\geq \frac{V}{2}$ special cols lead to 0

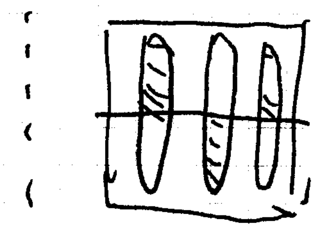


Limit to 0-bits

- Alice sends first bit



special colums



fix bit to 0 or 1

$u \to \frac{u}{2}, \quad v \to \frac{v}{2}$

(Use induction until whole truthtable is fixed)

Rectangle in end $\geq \left(\frac{V}{2^{a+b}} \text{ cols. w. } \frac{u}{2^d} \text{ ones} \right)$

$\geq \frac{u}{2^d} \times \frac{V}{2^{a+b}}$

$\square$

(Lemma gives strategy for proving lower bounds)

f balanced

- prove no big rectangle can be all 1's

Example: Set disjointness

$$U = [m] \times [n]$$

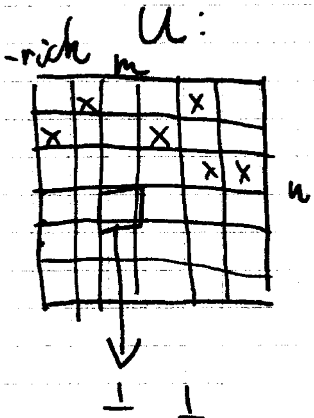
Alice	Bob
$X \subseteq U$	$Y \subseteq U$
$\lvert X \rvert = m$	$\lvert Y \rvert = n$

$f(X,Y) = 1$ iff. $X \cap Y = \emptyset$

Theorem

Either $a = \Omega(m \lg n)$
or $b = \Omega(n^{1-\varepsilon})$

Lem 1 (Richness) $[\Omega(n^m), m^n]$-rich



$\forall y$

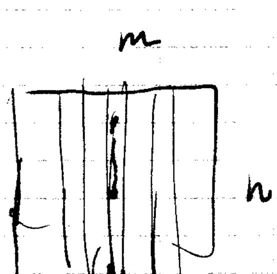
$\Pr_X [X \cap Y = \emptyset]$

Lem 2 $X \times Y$ $X \cap Y = \emptyset\ \forall x \in X, y \in Y$

impossible to have $\lvert X \rvert \geq$ __
and $\lvert Y \rvert \geq$

A's set

x_1 x_2 x_m



$X_i = \{x_i \mid x \in X\}$

$$|Y| \le m^{n-|X/i|} \cdot (m-1)^{|X_i|}$$

Partial Match

DB:

n points in $\{0,1\}^d$

query

$\{0,1,*\}^d$ "searching w. wildcards"

ex: 0100**0

Map $\{0,1,*\}^d \to \{0,1\}^{2d}$

$0 \to 01$

$1 \to 10$

$* \to 00$

query w. K *'s

$\lambda = K$ since $\|00 - 01\| = 1$ and $\|00 - 10\| = 1$

Disjointness $\to$ PM $\to$ NN

disjointness $\rightarrow$ PM <u>Reduction</u>:

$S \subseteq U \qquad |S| = m$

$T \subseteq U \qquad |T| = n$

$C : U \rightarrow \{0,1\}^{O(\lg n)}$

$\|C(x)\| = \|C(y)\|$

$x_1, x_2, \ldots, x_m \in S \qquad 0 \rightarrow 0$

$\qquad\qquad\qquad\qquad\quad 1 \rightarrow *$

ex.
7, 9, 13
110011   101011
00   *0*0**

$\forall y \in T, \ \forall i \in \{1, \ldots, m\} \qquad \overbrace{0000}^{(i-1)O(\lg n)} \, C(y) \, 0000$

$a = \Omega(n \lg n)$ or $b = \Omega(n^{1-\varepsilon})$

$a = \Omega(d)$ or $b = \Omega(n^{1-\varepsilon})$

time: $\Omega\left(\frac{d}{\lg S}\right)$

Miltersen, Nisan, et. al. STOC'95
Andoni, Indyk, Patrascu FOCS'06

Theorems: either $a = \Omega(\quad)$ or $b = \Omega(\quad)$

$\boxed{\text{alice}}$ $\boxed{\text{Bob}}$

w bits

$w = O(\lg n)$ $\quad t \lg s = \Omega(w)$

$= \Omega(\lg n)$

$t = \Omega(1)$

Patrascu, Thorup FOCS'06

$\boxed{\text{Alice}}$ $\quad k \lg s \quad (k \lg \frac{s}{k})$ $\quad \boxed{\text{Bob}}$

$X_1 \xrightarrow{\lg \binom{S}{k}}$ database

$\vdots \xleftarrow{kw}$

X_k

t rounds

$w = \lg n$

$a = \Omega(\lg n) \to a' = \Omega(k \lg n)$

$k = \frac{n}{\lg^{O(1)} n} \qquad k \lg \frac{S \lg^{O(1)} k}{n} = k \lg \lg n$

$\Rightarrow t = \Omega\left(\frac{\lg n}{\lg \lg n}\right)$

$\boxed{A}$ $\quad \boxed{B}$

$x \in X \quad y \in Y$

$\downarrow$

$f(x, y)$

$\rightsquigarrow$

$\boxed{A}$ $\quad \boxed{B}$

$(x_1, \dots, x_k) \in X^k \quad (y_1, \dots, y_k) \in Y^k$

$\downarrow$

$\wedge^k f = \wedge f_i(x_i, y_i)$

Lem: $\wedge^K f$ has 1-rect of size $(\alpha|X|)^K, (\beta|Y|)^K$

$\Rightarrow$ f has 1-rect of size $\alpha^3|X|, \beta^3|Y|$

Proof: $\mathcal{X}, \mathcal{Y} \subset X^K, Y^K$

At least $\frac{2}{3}K$ choices of i have $|\mathcal{X}_{/i}| \geq \alpha^3 |X|$

$|\mathcal{X}| = |X|^{K/3} \cdot (\alpha^3|X|)^{K/2} < \alpha^K |X|^K$

$|\mathcal{Y}_{/i}| \geq \beta^3 |Y|$ $\quad \cancel{\frac{1}{\alpha^K}}$

$\exists i$ where both ?

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f has $\Omega(1)$ density of 1's

$\wedge^K f$ in $[\Omega(|X|)^K, \Omega(|Y|)^K]$ - rich

$\hookrightarrow 2^{-O(K)}$

$$A, B \quad \left(\frac{\Omega(|X|)^K}{2^A}, \frac{\Omega(|Y|)^K}{2^{A+B}} \right)$$

$$\Downarrow$$

$$\frac{\Omega(|X|)}{2^{O(A)}}, \frac{\Omega(|Y|)}{2^{O([illegible]/K)}} \quad \text{for } f$$